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Nonperturbative model of Liouville gravity ★

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Abstract

We formulate nonperturbative 2D gravity in the framework of Liouville theory. In particular, we express the specific heat $\mathcal{Z}$ of pure gravity in terms of an expansion of integrals on moduli spaces of punctured Riemann spheres. We recognize the relevant divisors on moduli spaces and write the integrands in terms of the Liouville action. We evaluate the integrals (rational intersections) and show that $\mathcal{Z}$ satisfies the Painlevé I.

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1. Classical and quantum Liouville theory

The problem of formulating quantum Liouville theory is an open question which is related to crucial mathematical and physical aspects. For fixed genus some progress has been made in [1]. Nonperturbative results have been obtained in the framework of matrix models approach to noncritical strings [2] (see [3] for reviews).

In spite of these progresses, a nonperturbative formulation of quantum Liouville gravity in the continuum is still lacking. In this paper we solve the problem of finding a topological expansion of integrals involving the Liouville action such that it corresponds to the Painlevé I field. In doing this we will use important recent results in uniformization theory and algebraic geometry.

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The problems arising in the continuum formulation of Liouville gravity [1–4] are essentially:

(a) to evaluate Liouville correlators, in particular on Riemann surfaces of genus $h \geq 2$;
(b) to perform the integration on moduli spaces;
(c) to recover nonperturbative results from the topological expansion.

Results from matrix models and topological gravity show that these aspects are strictly related with the structure of $\mathcal{M}_{h,n}$, the moduli spaces of Riemann surfaces of genus h with n punctures. An important aspect of string theory is that the quantum geometry of strings is described by classical geometry of moduli sapces of Riemann surfaces. A similar aspect arises in quantum chaos on Riemann surfaces [5]. The basic reason for this interplay between quantum aspects and classical geometry is that determinants are described in terms of geodesic lengths by means of the Selberg trace formula. Since quantum Liouville theory should be described by some volume form on $\mathcal{M}_{h,n}$ and as the classical Liouville action is the Kähler potential for the natural (Weil–Petersson) metric on the moduli space, there should exist the correspondence:

Quantum Liouville theory
$\longrightarrow$ *Geometry of* $\mathcal{M}_{h,n}$
$\longrightarrow$ *Classical Liouville theory.*

In this context it should be emphasized that the classical Liouville action encodes a quantum feature such as regularization (see [6] for a discussion on this point). This may be related to the fact that for the canonical transformation that relates a particle moving in a Liouville potential to a free particle, the effective quantum generating function is identical to its classical conunterpart [7], so that there are no normal ordering problems.

Besides quantum Liouville theory also conformal field theories are intimately related with the geometry of moduli spaces. To explain this point let us denote by Σ a Riemann surface of genus h and consider the splitting of the measure on the metric

$$\mathcal{D}g = d[\mathbf{m}]\mathcal{D}_g v^z \mathcal{D}_g v^{\bar{z}} \mathcal{D}_g \sigma \det \nabla^z \det \nabla^{\bar{z}}, \tag{1.1}$$

where σ is the Liouville field, $d[\mathbf{m}]$ represents integration on moduli coordinates and v represent vector fields. Since

$$\|v, v\|^2_{g=e^\sigma \hat{g}} = \int_\Sigma \sqrt{\hat{g}}\hat{g}_{ab} e^{2\sigma} v^a v^b,$$

it follows that $\mathrm{Vol}_g(\mathit{Diff}(\Sigma))$ depends on σ. In critical string theory one usually *assumes* that this dependence can be absorbed into $\mathcal{D}_g\sigma$ and then drop the $\mathcal{D}_g v^z \mathcal{D}_g v^{\bar{z}}$ term. However this procedure has bot been fully investigated. Nevertheless the Liouville partition function should be an integration on the moduli space

$$Z_h = \int_{\mathcal{M}_h} d[\mathbf{m}] Z_h(\mathbf{m}). \tag{1.2}$$

In this context we stress that the connection between quantum Liouville theory, CFT and muduli spaces arises from the Mumford isomorphism [8]. To see this first notice that $Z_h(\mathbf{m})$ should be a well-defined volume form on $\mathcal{M}_h$. On the other hand the Mumford isomorphism is

$$\lambda_n \cong \lambda_1^{c_n}, \qquad c_n = 6n^2 - 6n + 1,$$

where $\lambda_n = \det \operatorname{ind} \bar{\partial}_n$ are the determinant line bundles. The fact that the metric measure cannot depend on the background choice implies that $c_{\text{tot}} = 0$. The Mumford isomorphism implies that $Z_h(\mathbf{m})$ is essentially the modulo square of a section of the bundle

$$\Lambda = \prod_{k=l}^{l} \lambda_k^{d_k}, \qquad \sum_{k=1}^{l} c_k d_k = 0, \tag{1.3}$$

where $-2c_j d_j$ is the central charge of the sector j. In the Polyakov string the matter and ghosts sectors have $d_1 = -\frac{1}{2}d$ and $d_2 = 1$, respectively, thus (1.3) gives for the Liouville sector $c_{\text{Liouv}} = 26 - d$.

An aspect related to the $d = 1$ barrier is that CFT matter of central charge d can be expressed in terms of a b–c system of weight n with $-2c_n = d$ [9]. The point is that since the maximum of $-2c_n$ is 1, this approach works for $d \leq 1$ only. The model is exactly a CFT realization of the Feigin–Fuchs approach where semi-infinite forms can be interpreted in terms of b–c system vacua. Of course one can use the bosonized version of the b–c system which is equivalent to the Coulomb gas approach.

For $d > 1$ it is not possible to represent the conformal matter in terms of a b–c system. In this case one can consider the β–γ system of weight n whose central charge is $2c_n$. However, the representation of the β–γ system in terms of free fields is a long-standing problem which seems related to the $d = 1$ barrier.

These aspects indicate that there is a connection between the barrier and the Mumford isomorphism. This is related to a similar structure considered in [10] in the framework of the geometrical formulation of 2D graviy [10,11] where representing elliptic and parabolic Liouville operators by means of a scalar field constrains the conformal matter to be in the sector $d \leq 1$.

The natural framework to investigate the aspects considered above is the theory of uniformization of Riemann surfaces where Liouville theory plays a crucial role. Actually, in [6] it has been shown that the Liouville action appears in the correlators (intersection numbers) of topological gravity [12]. The relationships between Liouville theory, matrix models and topological gravity suggest that it is possible to extend the above Liouville-topological gravity relationship by recovering the nonperturbative results of matrix models by continuum Liouville theory. In our model we will reduce all aspects concerning higher genus contributions to punctured Riemann spheres.

1.1. Reduction to punctured Riemann spheres

In [13] Knizhnik expressed the sum of the genus expansion as a CFT on an arbitrary N-sheet covering of the Riemann sphere with branch points. To each branch point he associated a vertex operator and proposed to express the infinite sum on all genus ($h \geq 2$) as the limit for $N \to \infty$ of a 'nonperturbative' partition function.

A natural way to get punctured spheres is by pinching all handles of a compact Riemann surface. Degenerate (singular) surfaces belong to the boundary of moduli spaces. These singularities play a fundamental role in the evaluation of relevant integrals (intersection theory). The fact that the *classical* Liouville action is the Kähler potential for the Weil–Petersson metric and the structure of the boundary of moduli space suggest to consider integrals on $\mathcal{M}_{h,n}$ in the framework of the Duistermaat–Heckman integration formula [14]. Thus the specific heat of quantum Liouville theory should be a sum of integrals Z_n^F on the moduli space of punctured Riemann spheres

$$\mathcal{M}_{0,n} = \frac{\{(z_1, \ldots, z_n) \in \hat{\mathbb{C}}^n \mid z_j \neq z_k \text{ for } j \neq k\}}{Symm(n) \times PSL(2, \mathbb{C})}$$

with the integrands involving the Liouville action. These remarks indicate that a theory à la Friedan–Shenker [15] can be concretely formulated to recover nonperturbative results in the continuum formulation. Actually, we will give the explicit realization of this formulation. We do not consider the above-mentioned points (a)–(c) separately, rather we will find the explicit form of the integrals $\mathcal{Z}_n^F$ on $\overline{\mathcal{M}}_{0,n}$ and recover nonperturbative results. Namely, we will show that

$$\mathcal{Z}(t) = t^{-12} \sum_{k=4}^{\infty} t^{5k} \int_{\overline{\mathcal{M}}_{0,k}} \left(\frac{\mathrm{i}\bar{\partial}\partial S_{\mathrm{cl}}^{(k)}}{2\pi^2} \right)^{k-4} \wedge \omega^{F_0} - \frac{t^3}{2} \tag{1.4}$$

satisfies the Painlevé I

$$\mathcal{Z}^2(t) - \tfrac{1}{3}\mathcal{Z}''(t) = t \tag{1.5}$$

so that, according to the results from matrix models, $\mathcal{Z}$ can be identified with the specific heat of pure gravity.

$S_{\mathrm{cl}}^{(k)}$ in (1.4) denotes the *classical* Liouville action on the k-punctured Riemann sphere. The class $[\omega^{F_0}]$ is the Poincaré dual of a divisor on the compactified moduli space $\overline{\mathcal{M}}_{0,k}$ which is given in terms of the $(2k-8)$-cycles defining the Deligne–Knudsen–Mumford boundary of $\overline{\mathcal{M}}_{0,k}$. The basic tools to obtain (1.4) are classical Liouville theory and intersection theory. The proof that $\mathcal{Z}(t)$ in (1.4) satisfies the Painlevé I is given in Section 5.

This result reproduces in the continuum the well-known results obtained in the matrix model approach to pure gravity [2]. The physical consequences of the model have been investigated in [16]. In particular, it turns out that the model corresponds to quantum Liouville theory with Einstein–Hilbert action having an imaginary part $\frac{1}{2}\pi$. In other words (1.4) corresponds to introduce a Θ-vacuum structure in the genus expansion [16]:

$$\mathcal{Z}(t) = \sum_{h=0}^{\infty} \int_{Met_h} \mathcal{D}g\, e^{-S(g)+\mathrm{i}(\Theta/2\pi)\int_\Sigma R\sqrt{g}}$$

$$=\sum_{h=0}^{\infty}(-1)^{1-h}\int\limits_{Met_h}\mathcal{D}g\,e^{-S(g)},\qquad \Theta=\tfrac{1}{2}\pi. \tag{1.6}$$

The effect of Θ-term is to convert the expansion into a series of alternating signs which is Borel summable.

An important point is that the specific heat of our model has a physical behaviour. According to standard thermodynamics, if one defines, following [2], the 'specific heat' as the second derivative of the free energy, it should be negative. In [16] it has been shown that the specific heat is negative for all $t>0$. Actually the standard choice [2] for the boundary condition in the asymptotic expansion is always positive for sufficiently large t. It seems that this choice is made in order to avoid an apparently unphysical behaviour such as the alternating sign of the asymptotic series. However, this 'unphysical behaviour' is only an effect of the perturbation expansion whereas the nonperturbative results are in complete agreement with basic physical principles. Thus the results of the model agree with standard thermodynamics and the theory is Borel summable. In our opinion, as emphasized in [16], the role of Θ-vacua is crucial for string theory in general. This aspect is related with the structure of the moduli space and to unitarity problems. To understand the relation between unitarity and the structure of moduli space one should consider that degenerated surfaces correspond to Feynman diagrams. The role of Θ-vacua should follow from a Feynman diagram analysis like applied to the string path-integral at the boundary of moduli spaces. We also notice that the presence of Θ-vacua should improve the convergence of the perturbation theory of critical strings. In other words one should expect that string perturbation theory with Θ-vacua converges.

Let us summarize the basic structures which will allow us to find a nonperturbative formulation in the continuum of a nontrivial quantum field theory such as 2D quantum gravity. We already noticed that points (a)–(c) considered above have been implicitly solved simultaneously in matrix models. It is natural to think that the same can be done in the continuum. However, in general it is technically very difficult to perform explicit integrations on moduli space. The fact that in the matrix models approach to 2D gravity relevant integrations on moduli spaces have been done implicitly suggests that the integrals are in some sense 'easy to compute'. Therefore one should expect that the relevant integrands on moduli spaces are a sort of total derivatives in such a way that the integrals receive contributions only from the boundary of moduli spaces. Due to the structure of the Deligne–Knudsen–Mumford compactification, one expects that once point (a) is solved, relevant integrals reduce to integrals on $\mathcal{M}_{0,n}$.

Let us summarize the main steps which we will consider in order to obtain Eqs. (1.4) and (1.5).

(1) First of all we observe that the Kähler potential for the Weil–Petersson volume form $\omega_{\mathrm{WP}}^{(n)}$ is the Liouville action evaluated on the classical solution [17].

(2) In [18], using the restriction phenomenon $[\omega_{\mathrm{WP}}^{(m)}]\doteq i^*[\omega_{\mathrm{WP}}^{(n)}]$, $n>m$ (see Appendix A), and computing intersection numbers between cycles on $\overline{\mathcal{M}}_{0,n}$, recursion relations for the Weil–Petersson volumes have been obtained.

(3) By a rescaling of the volumes one obtains recursion relations which are equivalent to a second-order nonlinear differential equation which is reminiscent of the Painlevé I. A crucial point is that the general structure of the recursion relations associated to this equation is unchanged if one considers integrals of deformed volume forms:

$$\omega_{\mathrm{WP}}^{(n)^{n-3}} \longrightarrow \omega_{\mathrm{WP}}^{(n)^{n-4}} \wedge \omega^F .$$

(4) As we will see, another crucial point is the fact that by a suitable choice of the two-form ω^F one obtains recursion relations which are equivalent to the Painlevé I. By deforming volumes one obtains in a natural way a solution for the specific heat of 2D gravity which has nonperturbative physical meaning.

Our results should be useful in order to properly understand the problem of the Liouville measure, to recover the determinants in (1.2) (i.e. the free field content of the theory) and the structure of the reduction mechanism. We also observe that our nonperturbative formulation of quantum Liouville theory should be useful in order to consider some nonperturbative aspects of other quantum field theories.

2. Reduction mechanism and Riemann surfaces with symmetries

The reduction to punctured Riemann spheres is particularly evident in topological field theory coupled to 2D gravity where higher genus contributions to the free energy $\langle 1\rangle_h$ can be written in terms of the sphere amplitudes of the puncture operator P [12,19]. The physical observables of the theory are the primary fields $\mathcal{O}_a$ ($\alpha = 0, 1, \dots, N-1$, $\mathcal{O}_0$ is the identity operator) and their gravitational descendents $\sigma_n(\mathcal{O}_\alpha)$, $n = 1, 2, \dots$ In the coupled system $\mathcal{O}_0$ becomes nontrivial and it is identified with P. Denoting by $\mathcal{L}_0$ the minimal Lagrangian, the more general one is $\mathcal{L} = \mathcal{L}_0 + \Sigma_{n,\alpha} t_{n,\alpha}\sigma_n(\mathcal{O}_\alpha)$, $\sigma_0(\mathcal{O}_\alpha) \equiv \mathcal{O}_\alpha$, where $t_{n,\alpha}$ are coupling constants. With this definition one can compute correlation functions with an insertion of σ_k just by differentiation $\langle 1\rangle_h$ with respect to t_k. Thus in general

$$\langle \sigma_{d_1}(\mathcal{O}_{\alpha_1}) \cdots \sigma_{d_n}(\mathcal{O}_{\alpha_n})\rangle_h = \frac{\partial}{\partial t_{d_1,\alpha_1}} \cdots \frac{\partial}{\partial t_{d_n,\alpha_n}} \langle 1\rangle_h .$$

Therefore $\langle 1\rangle_h$ is the crucial quantity to compute. By means of KdV recursion relations

$$\langle \sigma_1(P)P\rangle_h = 2\langle P^4\rangle_{h-1} + \frac{1}{2}\sum_{h'=0}^{h} \langle P^2\rangle_{h'}\langle P^2\rangle_{h-h'},$$

it is possible [19] to express $\langle 1\rangle_h$ as a sum of terms of the form $\langle p^{n_1}\rangle_0 \cdots \langle p^{n_j}\rangle_0 / \langle P^3\rangle_0^{h+j-1}$ for $1 \le j \le 3h-3$ with the constraint $\sum_{k=1}^{j} n_k = 3(j+h-1)$.

Reduction to punctured Riemann spheres arises also in the evaluation of $\mathrm{Vol}_{\mathrm{WP}}(\mathcal{M}_{h,n})$. Indeed, at least in some cases, there is a relationship between $\overline{\mathcal{M}}_{h,n}$, $\overline{\mathcal{M}}_{0,n+3h}$ and their volumes.[2] The first example is the geometric isomorphism [20] $\overline{\mathcal{M}}_{1,1} \cong \overline{\mathcal{M}}_{0,4}$, and

[2] The space $\mathcal{M}_{h,n}$ is not affine for $h > 2$. Conversely the space $\mathcal{M}_{0,k}$ is finitely covered by the affine space $V^{(k)}$ defined in (3.3). Thus for $h > 2$ they are not geometrical isomorphisms between $\overline{\mathcal{M}}_{h,n}$ and

$$\mathrm{Vol_{WP}}(\mathcal{M}_{1,1}) = 2\ \mathrm{Vol_{WP}}(\mathcal{M}_{0,4}). \tag{2.1}$$

To understand this result it is sufficient to recall that the $\wp$- function enters in the expression of the uniformizing connection of the once punctured torus $\Sigma_{1,1}$ (note that $\wp$ is a solution of the KdV equation)

$$T_{\Sigma_{1,1}} = \tfrac{1}{2}(\wp(\tau, z) + c(\tau)),$$

where $c(\tau)$ is accessory parameter for $\Sigma_{1,1}$. Eq. (2.1) follows from the fact that $T_{\Sigma_{1,1}}$ is related to the uniformizing connection $T_{\Sigma_{0,4}}$ of the Riemann sphere with four punctures since $\wp$ maps $\Sigma_{1,1}$ two-to-one onto the four punctured Riemann sphere. Let us notice that another isomorphism is [21] $\overline{\mathcal{M}}_{2,0} \cong \overline{M}_{0,6}$.

There is another way to understand why punctured spheres play a crucial role in 2D gravity. The point is to notice that equal size triangulated Riemann surfaces considered in matrix models can be realized in terms of thrice punctured spheres [22]. This aspect is related to arithmetic surfaces theory [22,23]. In this context one should investigate whether this kind of surface has some suitable symmetry to define antiholomorphic involution. This question is important in order to investigate Osterwalder–Schrader positivity. This is connected with the problem of defining the adjoint in higher genus that, on the sphere, can be done using the natural antinvolution $z \to \bar{z}^{-1}$. In higher genus this problem has been solved only on a Schottky double where there is a natural antinvolution [24]. Thus the Osterwalder– Schrader positivity problem is connected with the existence of automorphisms. In this context we note that the boundary components of moduli space have natural automorphisms, in particular curves with elliptic tails have the automorphism $x \to -x$ (see for example [25]). This aspect should be useful in considering the structure of the reduction mechanism.

Recently Harvey and González–Diéz [26] have considered loci of curves which are prime Galois covering of the sphere. In particular, they consider the important case of Riemann surface admitting nontrivial automorphisms and showed that there is a birational isomorphism between a subset of the moduli space $\mathcal{M}_h$ and $V^{(n)}$ (defined in (3.3)).

3. Weil–Petersson volumes and Liouville action

The relation between Liouville and uniformization theory of Riemann surfaces arises in considering the Liouville equation

$$\partial_{\bar{z}}\partial_z\varphi_{\mathrm{cl}} = \tfrac{1}{2}\mathrm{e}^{\varphi_{\mathrm{cl}}}, \tag{3.1}$$

which is uniquely satisfied by the Poincaré metric (i.e. the metric with Gaussian curvature -1). Let $H = \{w | \mathrm{Im}\, w > 0\}$ be the upper half-plane and Σ a Riemann surface with

$\overline{\mathcal{M}}_{0,n+3h}$. However, in principle, nothing exclude the possibility to express $\mathrm{Vol_{WP}}(\mathcal{M}_{h,n})$ in terms of $\mathrm{Vol_{WP}}(\mathcal{M}_{0,n+3h})$.

negative Euler characteristic (i.e. $\chi(\Sigma) = 2 - 2h - n < 0$). Since the Poincaré metric on H is $\mathrm{d}s^2 = (\mathrm{Im}w)^{-2}|\mathrm{d}w|^2$, we have

$$e^{\varphi_{\rm cl}} = \frac{|J_H^{-1\prime}|^2}{(\mathrm{Im}J_H^{-1})^2},$$

where J_H^{-1} is the inverse of the uniformizing map $J_H : H \to \Sigma$.

Let us introduce the n-punctured Riemann sphere $\Sigma_{0,n} = \hat{\mathbb{C}}\backslash\{z_1, \ldots, z_n\}$, $n \geq 3$,where $\hat{\mathbb{C}} \equiv \mathbb{C} \cup \{\infty\}$. Its moduli space is the space of classes of isomorphic $\Sigma_{0,n}$'s, that is

$$\mathcal{M}_{0,n} = \{(z_1, \ldots, z_n) \in \hat{\mathbb{C}}^n | z_j \neq z_k \text{ for } j \neq k\}/Symm(n) \times PSL(2, \mathbb{C}), \tag{3.2}$$

where $Symm(n)$ acts by permuting $z_1, \ldots, z_n$ whereas $PSL(2, \mathbb{C})$ acts by linear fractional transformations. By $PSL(2, \mathbb{C})$ we can recover the 'standard normalization': $z_{n-2} = 0, z_{n-1} = 1$ and $z_n = \infty$. Let us introduce the classical Liouville tensor or Fuchsian projective connection

$$T^F(z) = \{J_H^{-1}, z\} = \varphi_{\mathrm{cl}zz} - \tfrac{1}{2}\varphi_{\mathrm{cl}z}^2.$$

In the case of the punctured Riemann sphere we have

$$T^F(z) = \sum_{k=1}^{n-1}\left(\frac{1}{2(z - z_k)^2} + \frac{c_k}{z - z_k}\right),$$

where the coefficients $c_1, \ldots, c_{n-1}$, called *accessory parameters*, satisfy the constraint

$$\sum_{j=1}^{n-1} c_j = 0, \qquad \sum_{j=1}^{n-1} z_j c_j = 1 - \tfrac{1}{2}n.$$

These parameters are defined on the space

$$V^{(n)} = \{(z_1, \ldots, z_{n-3} \in \mathbb{C}^{n-3} | z_j \neq 0, 1; z_j \neq z_k, \text{ for } j \neq k\}. \tag{3.3}$$

Note that

$$\mathcal{M}_{0,n} \cong V^{(n)}/Symm(n), \tag{3.4}$$

where the action of $Symm(n)$ on $V^{(n)}$ is defined by comparing (3.2) with (3.4).

Let us now consider the compactification $\overline{V}^{(n)}$ in the sense of Deligne–Knudsen–Mumford [27]. The divisor at the boundary

$$D = \overline{V}^{(n)} \backslash V^{(n)},$$

decomposes in the sum of divisors $D_1, \ldots D_{[n/2]-1}$, which are subvarieties of real dimension $2n-8$. The locus D_k consists of surface that split, on removal of the node, into two Riemann spheres with $k+2$ and $n-k$ punctures. In particular, D_k consists of $C(k)$ copies of the space $\overline{V}^{(k+2)} \times \overline{V}^{(n-k)}$ where $C(k) = \binom{n}{k+1}$ for $k = 1, \ldots, \frac{1}{2}(n-3)$, n odd. In the case of n even the unique difference is for $k = \frac{1}{2}n - 1$, for which we have $C(\frac{1}{2}n - 1) =$

$\frac{1}{2}\binom{n}{\frac{1}{2}n}$. An important property of the divisors D_k's is that their image provides a basis in $H_{2n-8}(\overline{\mathcal{M}}_{0,n}, \mathbb{R})$.

In the case of the Riemann sphere with n-punctures $\Sigma_{0,n}$, Eq. (3.1) follows from the Liouville action [17]:

$$S^{(n)} = \lim_{r\to 0}\left[\int_{\Sigma^r_{0,n}} (\partial_z\varphi\partial_{\bar z}\varphi + e^{\varphi}) + 2\pi(n\log r + 2(n-2)\log|\log r|)\right],$$

where $\Sigma^r_{0,n} = \Sigma_{0,n}\backslash(\bigcup_{i=1}^{n-1}\{z||z-z_i|<r\}\cup\{z||z|>r^{-1}\})$. This action, evaluated on the classical solution, is the Kähler potential for the Weil–Petersson two-form on $V^{(n)}$ [17]:

$$\omega^{(n)}_{\mathrm{WP}} = \frac{\mathrm{i}}{2}\bar\partial\partial S^{(n)}_{\mathrm{cl}} = -\mathrm{i}\pi\sum_{j,k=1}^{n-3}\frac{\partial c_k}{\partial \bar z_j}\,\mathrm{d}\bar z_j\wedge \mathrm{d}z_k. \tag{3.5}$$

Let us consider the volume of moduli space of punctured Riemann spheres

$$\mathrm{Vol}_{\mathrm{WP}}(\mathcal{M}_{0,n}) = \frac{1}{(n-3)!}\int_{\overline{\mathcal{M}}_{0,n}} {\omega^{(n)}_{\mathrm{WP}}}^{n-3} = \frac{1}{(n-3)!}[\omega^{(n)}_{\mathrm{WP}}]^{n-3}\cap[\overline{\mathcal{M}}_{0,n}].$$

Recently it has been shown that [18]:

$$\mathrm{Vol}_{\mathrm{WP}}(\mathcal{M}_{0,n}) = \frac{1}{n!}\mathrm{Vol}_{\mathrm{WP}}(V^{(n)}) = \frac{\pi^{2(n-3)}V_n}{n!(n-3)!}, \quad n\ge 4,$$

where $V_n = \pi^{2(3-n)}[\omega^{(n)}_{\mathrm{WP}}]^{n-3}\cap[\overline{V}^{(n)}]$ satisfies the recursion relations:

$$V_3 = 1,$$
$$V_n = \frac{1}{2}\sum_{k=1}^{n-3}\frac{k(n-k-2)}{n-1}\binom{n}{k+1}\binom{n-4}{k-1}V_{k+2}V_{n-k}, \quad n\ge 4. \tag{3.6}$$

As we will see, the basic structures underlying Eq. (3.6) are classical Liouville theory and intersection theory

4. Volumes generating function

We now consider the differential equation associated with (3.6). In order to do this we note a crucial property of (3.6). Namely, defining

$$a_k = \frac{V_k}{(k-1)((k-3)!)^2}, \quad k\ge 3, \tag{4.1}$$

(3.6) assumes the simple form

$$a_3 = \tfrac{1}{2} \qquad a_n = \frac{1}{2}\frac{n(n-2)}{(n-1)(n-3)}\sum_{k=1}^{n-3}a_{k+2}a_{n-k}, \quad \ge 4. \tag{4.2}$$

Defining the generating function for Weil–Petersson volumes

$$g(t) = \sum_{k=3}^{\infty} a_k t^{k-1}, \tag{4.3}$$

one can check that Eq. (4.2) implies that the function g satisfies the differential equation

$$g'' = \frac{g'^2 t - gg' + g't}{t(t-g)}. \tag{4.4}$$

4.1. Volume deformation

We now consider the deformation of the Weil–Petersson volume form. First of all note that by (3.5) and (4.3)

$$g(t) = \sum_{k=3}^{\infty} \frac{k(k-2)}{(k-3)!} t^{k-1} \int\limits_{\overline{\mathcal{M}}_{0,k}} \left(\frac{\mathrm{i}\bar{\partial}\partial S_{cl}^{(k)}}{2\pi^2} \right)^{k-3}, \tag{4.5}$$

where '$\int_{\overline{\mathcal{M}}_{0,3}} 1$'$\equiv \frac{1}{6}$. Function $g(t)$ resembles a topological expansion of string theory. Furthermore, the structure of Eq. (4.4) suggests that with a suitable deformation of the volume form it should be possible to obtain the Painlevé I. These remarks indicate that it is possible to recover the specific heat of pure gravity in the continuum. Actually, we will recover the Painlevé I by classical Liouville theory. In particular, we will obtain the recursion relations for the Painlevé I by performing a suitable deformation of the Weil–Petersson volume form $\omega_{\mathrm{WP}}^{(n)^{n-3}}$. Remarkably, as we will show, it is possible to perform the substitution

$$\omega_{\mathrm{WP}}^{(n)^{n-3}} \longrightarrow \omega_{\mathrm{WP}}^{(n)^{n-4}} \wedge \omega^F \tag{4.6}$$

in (4.5) without changing the general structure of (4.2); that is we will obtain recursion relations of the following structure:

$$A_n = C(n) \sum_{k=1}^{n-3} A_{k+2} A_{n-k}, \quad n \geq 4. \tag{4.7}$$

We stress that this is a crucial point in our construction.

The first problem now is to find a suitable expansion for the Painlevé I field such that the structure of the associated recursion relation be the same of (4.7). Remarkably this expansion exists, namely

$$f(t) = t^{-12} \sum_{k=3}^{\infty} d_k t^{5k}. \tag{4.8}$$

It is interesting that in searching the expansion reproducing the general structure of (4.2), which is a result obtained from continuous Liouville theory, one obtains an expansion involving only *positive* powers of t. With this expansion the Painlevé I

$$f^2(t) - \tfrac{1}{3} f''(t) = t \tag{4.9}$$

is equivalent to the recursion relations [3]

$$d_n = \frac{3}{(12-5n)(13-5n)} \sum_{k=1}^{n-3} d_{k+2} d_{n-k}, \qquad d_3 = -\tfrac{1}{2}, \tag{4.10}$$

which has the same structure of (4.2). Note that expansion (4.8) corresponds to the initial conditions $f(0) = f'(0) = 0$.

We now find the volume form reproducing (4.10). To understand which kind of modification to $\omega_{\mathrm{WP}}^{(n)^{n-3}}$ can be performed without changing the basic structure of (4.2) we recall the main steps in [18] to obtain (3.6). As we will see, Eqs. (4.11) and (4.13) below (which will be derived in the Appendix A) are crucial steps to compute Weil–Petersson volumes and their deformation.

Let D_{WP} be the $(2n-8)$-cycle dual to the Weil–Petersson class $[\omega_{\mathrm{WP}}^{(n)}]$. To compute the volumes it is useful to expand D_{WP} in terms of the divisors D_k in the boundary of the moduli space. It turns out that [18]:

$$D_{\mathrm{WP}} = \frac{\pi^2}{n-1} \sum_{k=1}^{[n/2]-1} k(n-k-2) D_k. \tag{4.11}$$

Let us set $D^{(n)} = D_{\mathrm{WP}}/\pi^2$ and $\omega^{(n)} = \omega_{\mathrm{WP}}^{(n)}/\pi^2$ so that $[\omega^{(n)}] \in H^2(\overline{\mathcal{M}}_{0,n}\mathbb{Q})$. We now consider

$$V_n = [\omega^{(n)}]^{n-3} \cap [\overline{V}^{(n)}] = [\omega^{(n)}]^{n-4} \cap ([\omega^{(n)}] \cap [\overline{V}^{(n)}])$$

and notice that since

$$[\omega^{(n)}] \cap [\overline{V}^{(n)}] = D^{(n)} \cdot \overline{V}^{(n)} = D^{(n)},$$

it follows that

$$V_n = [\omega^{(n)}]^{n-4} \cap [D^{(n)}] = \frac{1}{n-1} \sum_{k=1}^{[n/2]-1} k(n-k-2)[\omega^{(n)}]^{n-4} \cap [D_k].$$

Since D_k consists of $C(k)$ copies of space $\overline{V}^{(k+2)} \times \overline{V}^{(n-k)}$, we have

$$V_n = \frac{1}{n-1} \sum_{k=1}^{[n/2]-1} k(n-k-2) C(k) [\omega^{(n)}]^{n-4} \cap [\overline{V}^{(k+2)} \times \overline{V}^{(n-k)}] \tag{4.12}$$

that by [18]

$$\begin{aligned} &[\omega^{(n)}]^{n-4} \cap [\overline{V}^{(k+2)} \times \overline{V}^{(n-k)}] \\ &\quad = [\omega^{(k+2)} + \omega^{(n-k)}]^{n-4} \cap [\overline{V}^{(k+2)} \times \overline{V}^{(n-k)}], \\ &\quad = \binom{n-4}{k-1} ([\omega^{(k+2)}]^{k-1} \cap \overline{V}^{(k+2)})([\omega^{(n-k)}]^{n-k-3} \cap \overline{V}^{(n-k)}), \end{aligned} \tag{4.13}$$

coincides with (3.6).

[3] Notice that $(-1)^k d_k$ is positive.

5. Liouville F-models and 2D gravity

A crucial point is that there exist a systematic way to deform the volume form without changing the general structure of the associated recursion relations. We note that our method should be useful to investigate also some general algebraic-geometrical structure and some aspects concerning nonlinear differential equations.

Let us introduce the divisor

$$D^F = \frac{1}{n-1} \sum_{k=1}^{[n/2]-1} k(n-k-2)F(n,k)D_k, \tag{5.1}$$

where $F(n,k)$ is a function to be determined. Let $[\omega^F]$ be the Poincaré dual to D^F and define

$$Z_n^F = \int_{\overline{\mathcal{M}}_{0,n}} \omega^{(n)^{n-4}} \wedge \omega^F = \int_{\overline{\mathcal{M}}_{0,n}} \left(\frac{\mathrm{i}\bar\partial\partial S_{cl}^{(n)}}{2\pi^2}\right)^{n-4} \wedge \omega^F, \quad n \geq 4. \tag{5.2}$$

A basic point is that we can use the recursion relation (4.2) to evaluate (5.2) and obtain nonperturbative results. This possibility is based on the fact that

$$[\omega^{(n)}]^{n-3} \cap [\overline{V}^{(n)}] = [\omega^{(n)}]^{n-4} \cap [D^{(n)}],$$

implying that the general structure of (4.2) (the same of (4.10)) is unchanged under the substitution

$$\omega^{(n)^{n-3}} \longrightarrow \omega^{(n)^{n-4}} \wedge \omega^F. \tag{5.3}$$

To see this note that

$$\begin{aligned} Z_n^F &= \frac{1}{n!}[\omega^{(n)}]^{n-4} \cap [D^F] \\ &= \frac{1}{(n-1)n!} \sum_{k=1}^{[n/2]-1} F(n,k)k(n-k-2)[\omega^{(n)}]^{n-4} \cap [D_k]. \end{aligned} \tag{5.4}$$

On the other hand by (4.13)

$$\begin{aligned} &\sum_{k=1}^{[n/2]-1} F(n,k)k(n-k-2)[\omega^{(n)}]^{n-4} \cap [D_k] \\ &\quad = \frac{1}{2}\sum_{k=1}^{n-3} F(n,k)k(n-k-2)\binom{n}{k+1}\binom{n-4}{k-1} V_{k+2}V_{n-k}, \end{aligned} \tag{5.5}$$

and by (4.1)

$$Z_n^F = \frac{(n-4)!}{2(n-1)} \sum_{k=1}^{n-3} F(n,k)a_{k+2}a_{n-k}, \quad n \geq 4. \tag{5.6}$$

Let us define the 'Liouville F-models'

$$\mathcal{Z}^{F,\alpha}(x) = x^{-\alpha}\sum_{k=3}^{\infty} x^k Z_k^F, \tag{5.7}$$

where x is the coupling constant. These models are classified by α, $F(n,k)$ and Z_3^F.

5.1. Pure gravity: explicit solution

We now show that $\mathcal{Z}^{F,\alpha}(x)$ includes pure gravity. In fact, putting

$$\mathcal{Z}(t) = \mathcal{Z}^{F_0,\alpha_0}(t^5), \quad Z_3^{F_0} = -\tfrac{1}{2}, \quad \alpha_0 = \tfrac{12}{5}, \tag{5.8}$$

where

$$F_0(n,k) = \frac{6(n-1)}{(12-5n)(13-5n)(n-4)!}\,\frac{Z_{k+2}^{F_0} Z_{n-k}^{F_0}}{a_{k+2}a_{n-k}} \tag{5.9}$$

we have, by (5.6) and (5.9),

$$Z_3^{F_0} = -\tfrac{1}{2},$$
$$Z_n^{F_0} = \frac{3}{(12-5n)(13-5n)}\sum_{k=1}^{n-3} Z_{k+2}^{F_0} Z_{n-k}^{F_0}, \quad n \geq 4 \tag{5.10}$$

so that by (4.8)–(4.10)

$$\mathcal{Z}(t) = t^{-12}\sum_{k=4}^{\infty} t^{5k}\int_{\overline{\mathcal{M}}_{0,k}} \left(\frac{\mathrm{i}\bar{\partial}\partial S_{cl}^{(k)}}{2\pi^2}\right)^{k-4} \wedge \omega^{F_0} - \frac{t^3}{2} \tag{5.11}$$

satisfies the Painlevé I

$$\mathcal{Z}^2(t) - \tfrac{1}{3}\mathcal{Z}''(t) = t \tag{5.12}$$

with initial conditions $\mathcal{Z}(0) = \mathcal{Z}'(0) = 0$. We observe that other initial conditions can be reproduced by a suitable choice of $F(n,k)$.

6. Outlook

In conclusion we have introduced a class of Liouville models by defining a suitable D^F divisor which is a deformation of the Weil–Petersson divisor parameterized by α, $F(n,k)$ and Z_3^F. These Liouville F-models include pure gravity. In this context we recall that the Liouville action arises also in the correlators of topological gravity [6].

We note that our results should be useful in solving problems (a) and (b) considered in Section 1 (point (c) is solved by (5.11) and (5.12)). In particular, connecting the coefficients of the local expansion (5.11) with the related coefficients of the asymptotic expansion derived in [16], one obtains a relation between the genus h contribution to the free energy (related to the Liouville density $Z_h(\mathbf{m})$ in (1.2)) and Z_n^F. This formula should be also useful to clarify the details of the mechanism which allows us to reduce integrals on the moduli space of higher genus Riemann surfaces to integrals on the moduli space of punctured Riemann spheres. In this context we note that the divisor associated to the specific heat in the asymptotic (genus) expansion has been found in [28]. By investigating the structure of this divisor, it should be possible to recover the field content of the Liouville path-integral.

Let us comment about a possible related nonperturbative formulation on the upper half-plane H (see also [6]). Punctures on the Riemann sphere correspond to real points $J_H^{-1}(z_k)$ on the boundary of H. Correspondingly one can define hyperelliptic Riemann surfaces. In the case of infinite genus one can apply McKean–Trubowitz theory [29] which is related to matrix models. This suggests a nonperturbative formulation on H with the image of punctures related to the eigenvalues of the Hermitian matrix models. In some discrete version of this approach one should be able to connect this formulation with the ideas at the basis of [30].

Finally, we note that the algebraic-geometrical approach to 2D gravity considered in this paper is related to other interesting aspects such as anyon theory [31], compactification of configuration spaces [32] and quantum cohomology [33]. We also observe that Eq. (4.4) has been essentially solved in [34].

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Appendix A. Weil–Petersson divisor

Let us start by proving the restriction phenomenon. Namely we show that from the natural embedding

$$i : \overline{V}^{(m)} \to \overline{V}^{(m)} \times * \to \overline{V}^{(m)} \times \overline{V}^{(n-m+2)} \to \partial\overline{V}^{(n)} \to \overline{V}^{(n)}, \quad n > m,$$

where $*$ is an arbitrary point in $\overline{V}^{(n-m+2)}$, it follows that [20,35]:

$$[\omega_{\mathrm{WP}}^{(m)}] = i^*[\omega_{\mathrm{WP}}^{(n)}], \quad n > m, \tag{A.1}$$

which has been used in Eq. (4.13). In order to prove (A.1) we need to consider the Fenchel–Nielsen parametrization of the Teichmüller space (see for example [36]). Let $\{P_i\}$ be a set of surfaces homeomorphic to $\hat{\mathbb{C}}$ minus three open discs. Each P_i has a hyperbolic structure with geodesic boundary whose length may be arbitrarily prescribed in the interval $[0, \infty)$ (a length 0 corresponds to a puncture). Let

$$\Sigma_{0,n} = \mathbb{C}\backslash\{z_1, \ldots, z_{n-3}, 0, 1\},$$

be a genus 0 surface with $n \geq 3$ punctures. It can be obtained by glueing $\{P_i\}_{i=1,\dots,n-2}$ identifying the different boundary components in $n-3$ geodesics on $\sum_{0,n}$. Clearly, to completely characterize the glueing procedure, we need also to distinguish twisted boundary components. To this end, for each geodesic α_i we denote by τ_i the coordinate describing twists from an arbitrary reference position. Denoting by $l_i = l_{\alpha_i}$ the length of each geodesic, we define the Fenchel–Nielsen form

$$\omega_{\mathrm{FN}}^{(n)} = \sum_{j=1}^{n-3} \mathrm{d}l_j \wedge \mathrm{d}\tau_j = \sum_{j=1}^{n-3} l_j \,\mathrm{d}l_j \wedge \mathrm{d}\theta_j,$$

where θ_j is the twisting angle (it has been proved that $\omega_{\mathrm{FN}}^{(n)}$ does not depend on the particular geodesical dissection of $\Sigma_{0,n}$).

The observation that the smooth reduction of $\omega_{\mathrm{FN}}^{(n)}$ to $\partial\overline{V}^{(n)}$ is performed letting one or more geodesical lengths go to 0 giving a well-defined geodesical dissection, implies $[\omega_{\mathrm{FN}}^{(m)}] = i^*[\omega_{\mathrm{FN}}^{(n)}]$, $n > m$. Eq. (A.1) follows by noticing that [20,35]:

$$[\omega_{\mathrm{FN}}^{(n)}] = [\omega_{\mathrm{WP}}^{(n)}]$$

in $H^2(\overline{V}^{(n)}, \mathbb{R})$.

Now, following [18], we derive (4.11). Let us consider the embedding of $\Sigma_{0,n-k}$ in $V^{(n-k+1)}$

$$\Sigma_{0,n-k} \longrightarrow V^{(n-k+1)}, \qquad z \mapsto (z_1, \dots, z_{n-k-3}, z) \in V^{(n-k+1)}, \quad z \in \Sigma_{0,n-k}.$$

Observer that $\Sigma_{0,n-1}$ embeds into $\overline{V}^{(n)}$. There exists a natural embedding in $\overline{V}^{(n)}$ also for the surfaces $\Sigma_{0,n-k}$, $k = 2, \dots, [\frac{1}{2}n] - 1$, namely

$$\Sigma_{0,n-k} \to V^{(n-k+1)} \to \overline{V}^{(k+1)} \times \overline{V}^{(n-k+1)} \to D_{k-1} \to \overline{V}^{(n)}, \quad k = 2, \dots, \left[\tfrac{1}{2}n\right] - 1. \tag{A.2}$$

The closure of the image of $\Sigma_{0,n-k}$ in $\overline{V}^{(n)}$ defines a 2-cycle C_k isomorphic to $\hat{\mathbb{C}}$. By (A.1) and (A.2) it follows that

$$[\omega_{\mathrm{WP}}^{(n)}] \cap [C_k] = \int_{\mathrm{i}\Sigma_{0,n-k}} \omega_{\mathrm{WP}}^{(n)} = \int_{\Sigma_{0,n-k}} i^*\omega_{\mathrm{WP}}^{(n)} = \int_{\Sigma_{0,n-k}} \omega_{\mathrm{WP}}^{(n-k+1)}, \tag{A.3}$$

where $\cap$ denotes the topological cap product. Note that $[\omega_{\mathrm{WP}}^{(n)} \cap [C_k] = D_{\mathrm{WP}} \cdot C_k$ where $\cdot$ denotes the topological interesection (see for example [37]).

To perform the last integral we use (3.5) and the asymptotic behaviour of the classical Liouville action when the punctures coalesce [38]:

$$\partial_{z_i} S_{\mathrm{cl}}^{(n)}(z_1, \dots, z_{n-3}) = \begin{cases} \dfrac{\pi}{z_i - z_k} + \mathrm{O}\left(\dfrac{1}{|z_i - z_k|}\right), & z_i \to z_k, \quad k \neq n, \\ \dfrac{\pi}{z_i} + \mathrm{O}\left(\dfrac{1}{|z_i|}\right), & z_i \to \infty. \end{cases} \tag{A.4}$$

We have [4]

$$D_{\mathrm{WP}} \cdot C_k = \int_{\mathbb{C}} \omega_{\mathrm{WP}}^{(n-k+1)} = -\frac{1}{2\mathrm{i}} \lim_{r\to 0} \int_{\mathbb{C}\setminus\Delta_r} d\partial_{z_{n-k-2}} S_{\mathrm{cl}}^{(n-k+1)}\, \mathrm{d}z_{n-k-2},$$

where Δ_r is the union of $n-k-1$ disks of radius r centred at $z_1, \ldots, z_{n-k-3}, 0, 1$.

Let us set $z \equiv z_{n-k-2}$. Since $\partial_z S_{\mathrm{cl}}^{(n-k+1)} \in C^\infty(\mathbb{C}\setminus\Delta_r)$, we can apply Stokes theorem

$$\int_{\mathbb{C}\setminus\Delta_r} d\partial_z S_{\mathrm{cl}}^{(n-k+1)}\, \mathrm{d}z$$

$$= \int_{\partial\mathbb{C}} \partial_z S_{\mathrm{cl}}^{(n-k+1)}\, \mathrm{d}z - \int_{\partial\Delta_r} \partial_z S_{\mathrm{cl}}^{(n-k+1)}\, \mathrm{d}z = 2\mathrm{i}\pi^2 - 2\mathrm{i}\pi^2(n-k+1).$$

On the other hand

$$\lim_{r\to 0} \int_{\Delta_r} d\partial_z S_{\mathrm{cl}}^{(n-k+1)}\, \mathrm{d}z = 0$$

so that

$$D_{\mathrm{WP}} \cdot C_k = \pi^2(n-k-2). \tag{A.5}$$

Eq. (4.11) follows immediately from (A.5) and from the nonsingular matrix $A_{jk} = C_j \cdot D_k$ of intersection numbers between the 2-cycles C_j and the $(2n-8)$-cycles D_k [18]

$$A = \begin{pmatrix} n-1 & 0 & 0 & 0 & \ldots & 0 & 0 \\ n-4 & 1 & 0 & 0 & \ldots & 0 & 0 \\ n-4 & -1 & 1 & 0 & \ldots & 0 & 0 \\ n-5 & 0 & -1 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ n-[n/2] & 0 & 0 & 0 & \ldots & -1 & 1 \end{pmatrix}, \tag{A.6}$$

where $C_j \cdot D_1 = n-j-1$ for $j \geq 4$.

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